near these points and the stresses in these zones are redistributed and smoothed out. The singularities are caused by the fact that the contour γ reaches the free boundary and they represent a case different from the case in which the region occupied by the inclusion is internal.

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ON THE PROOF OF THE SAINT-VENANT PRINCIPLE FOR BODIES OF ARBITRARY SHAPE

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Toupin [1] proved that the stresses in a cylindrical rod, caused by application of a self-equalizing load at the endface, decrease exponentially with distance from the endface. An estimate has been obtained for a constant in the exponential in terms of the smallest natural vibrations frequency of an elastic cylinder.

A determination of the energy decay rate is given below for bodies of arbitrary shape and its estimate is given in terms of some characteristics of the body geometry, including the Poincaré and Korn constants of the cross section. These constants are known in the case of a circular rod and the estimate is given in numbers.

The dependence of the energy decay rate on the body shape is examined. It is shown that for cone-type bodies a power-law estimate holds for the energy decay which goes into an exponential estimate in the limit as the cone degenerates into a cylinder. Analogous estimates for the stresses result from the estimates for the energy.

1. Determination of the energy decay rate. Within the framework of a geometrically linear theory, let us generally consider an inhomogeneous, anisotropic and physically nonlinear elastic solid (see [2]). We refer the undeformed state of the solid to a Cartesian coordinate system $x^\circ \equiv x, x^\alpha$ (the Greek superscripts α, β, γ, . . . take on the values 1, 2).

Let the part of the solid in the half-plane $x > 0$ be load-free and let the state of stress be caused by some external effects on the part of the solid in the half-plane $x \leqslant 0$. Further we will study the parameters independent of the nature of these effects, therefore without limiting the generality, it can be assumed that the deformation of the body in the $x \geqslant 0$ half-plane is caused by some surface forces applied in a section of the solid by the $x = 0$ plane.

Let $V(x)$ denote a set of points of the body with abscissa greater than x, $\Omega(x)$ is a section of the body by the plane $x = \text{const}$ which separates $V(x)$ from the rest of the body (see Fig. 1), U is the elastic energy per unit volume, and E is the elastic energy of the body $V(x)$

$$E = \int_{V(x)} U dx^1 dx^2 dx$$

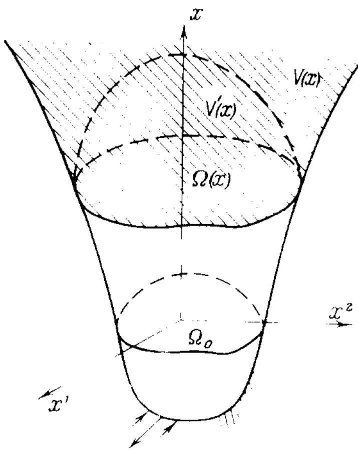

Fig. 1

It is assumed that the domain $\Omega(x)$ is bounded for each x. Let the elastic body $V(x)$ be deformed by surface forces p^i applied at Ω and let the appropriate density of the elastic energy U_p be calculated for each field of surface forces as a result of solving the elasticity theory problem. The subscript p emphasizes that this density depends on the applied load.

Let us consider the minimal value of the ratio between the "surface" and volume elastic energies

$$\gamma = \inf_{p^i} \left(\int_{\Omega(x)} U_p dx^1 dx^2 \Big/ \int_{V(x)} U_p dx^1 dx^2 dx \right) \tag{1.1}$$

Here the minimum is evaluated over all possible values of the surface load p^i. The constant γ evidently depends only on the geometry of the domain $V(x)$ and the elastic constants, and has the dimensionality (length) $^{-1}$. If the position of the body relative to the coordinate system is fixed, then γ becomes a function of x. The function $\gamma(x)$ determines the rate of elastic energy decay. This results from the following assertion.

Let an elastic body $V_0 \equiv V(0)$ be deformed by self-equilibrated surface forces applied at $\Omega_0 \equiv \Omega(0)$. Then the estimate

$$E(x) \leqslant E(0) \exp\left(-\int_0^x \gamma(x)\, dx\right) \tag{1.2}$$

holds for the elastic energy $E(x)$ of the part $V(x)$ of the body V_0.

In fact it follows from (1.1) that for any x

$$\gamma(x) E(x) \leqslant \int_{\Omega(x)} U dx^1 dx^2 \tag{1.3}$$

Using the formula

$$\frac{dE}{dx} = -\int_{\Omega} U dx^1 dx^2 \tag{1.4}$$

the relationship (1.3) can be rewritten as the differential inequality

$$\gamma(x) E(x) + \frac{dE(x)}{dx} \leqslant 0$$

from which (1.2) follows.

N o t e s. 1. In the case of a semi-infinite homogeneous elastic cylinder, the constant γ is evidently determined completely by the geometry of the cylinder cross section and by the elastic constants and is independent of x. Hence

$$E(x) \leqslant E(0)\, e^{-\gamma x} \tag{1.5}$$

2. The estimate in [1] which corresponds to (1.5) is of the form

$$E(x) \leqslant E(0) \exp\left[-\frac{x-l}{s(l)}\right], \qquad s(l) = \left(\frac{\mu}{\rho\omega_0^2(l)}\right)^{1/2} \tag{1.6}$$

$$\mu^* = \frac{\mu_M^2}{\mu_m}$$

Here $\omega_0(l)$ is the smallest natural vibrations frequency of a cylinder of hight l, μ_M and μ_m are the maximum and minimum elastic moduli, respectively, ρ is the density and l is an arbitrary length not exceeding half the length of the rod. It was used implicitly that $x > l$ in deriving (1.6). The inequalities (1.2), (1.5) hold for any $x \geqslant 0$.

3. It is shown in [1] that a point-by-point estimate for the stresses results from the estimate for the elastic energy, hence, it is sufficient to examine the question of the rate of elastic energy decay. The rate of stress decay for a linearly elastic body is half the rate of elastic energy decay.

4. The selection of the planes $x = \text{const}$ as the family of surfaces $\Omega(x)$ is not essential. Any other family of surfaces separating the domain of load application from the rest of the body. For example, in calculating the decay rate of stresses caused by the self-balanced load applied to the compact section of the half-space boundary, it is natural to take a hemisphere as the surface $\Omega(x)$. The choice of the surface is dictated by the possibility of obtaining an estimate for γ.

In the terminology defined above, the crux of the Saint Venant principle is the assertion that γ differs from zero and is "not too small". The "proof" of the Saint Venant principle reduces to constructing "good enough" lower bounds for γ.

A more accurate formulation of the Saint Venant principle encounters difficulties associated with the introduction of the "smallness criterion" for γ. For example, the condition $\varepsilon L^{-1} < \gamma$ is substantially taken as the "smallness criterion" for rods, where L is the diameter of the rod cross section, and ε is a number on the order of unity. Meanwhile, it can be shown that γ can be made arbitrarily small for a rod of fixed diameter because of the choice of the cross section.

Similar refuting examples exist for all known formulations of the Saint Venant principle. When the quantitative characteristics of stress decay of the type γ are studied, the need for any heuristic formulations understandably drops out.

In connection with the above, a number of questions arises.

1) What is the actual energy (or stress) decay rate in cylindrical rods with the simplest cross-sectional shapes ?

2) How can the energy decay rate in a cylindrical rod be estimated in terms of a parameter more accessible than the natural vibrations frequency, in terms of the geometric characteristics of the cross section, for example (*) ?

3) How can the energy decay rate be estimated in bodies whose geometric shape contains more arbitrary parameters than in a rod, for example, in bodies of cone type ?

4) What is the energy decay rate in thin bodies of the plate and shell type ?

5) What is the "energy" decay rate for arbitrary elliptical systems (**) ? (Footnote (**) at the next page).

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*) An example of this kind of estimate might be the estimate of the torsional stiffness of a rod, the electrostatic capacity of a body, and the fundamental natural frequency of membrane vibrations obtained in [3].

This area of questions has hardly been studied. Even the energy decay rate in a circular rod is not known. Particular results have been obtained recently in [4–7]. The results of this work refer to questions (2) and (3).

2. Assumptions relative to elastic energy. Let $w \equiv w^0$, w^α be projections of the displacement on the axes x, x^α, and let ε_{ij} be the strain tensor components: $\varepsilon_{ij} = w_{(i,j)}$. The parentheses in the subscripts here denote the symmetrization operation, and the comma denotes differentiation with respect to x^i, and the Latin subscripts take on the values 0, 1, 2. The zero super- or subscript is ordinarily omitted in writing the vector and tensor components, so that $w^\circ \equiv w$, $\varepsilon_{0\alpha} \equiv \varepsilon_\alpha$, $\varepsilon_{00} \equiv \varepsilon$, etc.

It is convenient to consider the elastic energy per unit volume U and the stress tensor components p^{ij} dimensionless by referring them to the shear modulus μ (μ is any of the elastic moduli for an anisotropic body). The elastic energy U for each particle of a body with the coordinates x^i is a convex, differentiable function of the strain tensor components by assumption, $U = U(x^i, \varepsilon_{ij})$.

Without limiting the generality, it can be assumed that

$$U(x^i, 0) = 0, \qquad \partial U / \partial \varepsilon_{ij}|_{\varepsilon_{ij}=0} = 0 \tag{2.1}$$

From the convexity of U in ε_{ij} there follows the inequality

$$U \leqslant \varepsilon_{ij} \partial U / \partial \varepsilon_{ij} \tag{2.2}$$

Let us assume that the elastic energy U can have quadratic forms in the strain tensor components as upper and lower bounds

$$\begin{gathered} {}^1/_2\,(A_1 \varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} + A_2 \varepsilon^2 + 2A_3 \varepsilon_\alpha \varepsilon^\alpha) \leqslant U(x^i, \varepsilon_{ij}) \leqslant \\ {}^1/_2\,(B_1 \varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} + B_2 \varepsilon^2 + 2B_3 \varepsilon_\alpha \varepsilon^\alpha) \end{gathered} \tag{2.3}$$

where $A_1, \ldots, B_3$ are constants independent of the coordinates.

Let us introduce the function $U^*(x^i, p^{ij})$, the Young transform of the function $U(x^i, \varepsilon_{ij})$ in the variable ε_{ij}

$$U^*(x^i, p^{ij}) = \sup_{\varepsilon_{ij}} [p^{ij}\varepsilon_{ij} - U(x^i, \varepsilon_{ij})] \tag{2.4}$$

It follows from (2.3) and (2.4)

$$\begin{gathered} {}^1/_2\,(B_1^{-1} p_{\alpha\beta} p^{\alpha\beta} + B_2^{-1} p^2 + 2B_3^{-1} p_\alpha p^\alpha) \leqslant U^*(x^i, p^{ij}) \leqslant \\ {}^1/_2 (A_1^{-1} p_{\alpha\beta} p^{\alpha\beta} + A_2^{-1} p^2 + 2A_3^{-1} p_\alpha p^\alpha) \end{gathered} \tag{2.5}$$

In particular, if U is a quadratic form in ε_{ij}, then substitution of the expressions for the strain tensor components ε_{ij} in terms of the stress tensor components $p^{ij} = \partial U / \partial \varepsilon_{ij}$ into U results in the relationship $U(x^i, \varepsilon_{ij}(p^{kl})) = U^*(x^i, p^{ij})$, and (2.5)

(**) It is clear that the Saint Venant principle holds not only in elasticity theory. For example, the following changes must be made in the preceding exposition for the potential flow of an ideal incompressible fluid: elastic energy → kinetic energy, stress → velocity, self-balanced load in Ω → fluid discharge through Ω equal to zero.

yields an estimate of the elastic energy in terms of the quadratic forms in the stress tensor components.

Let B denote the constant in the inequality

$$p^2 + p_\alpha p^\alpha \leqslant BU \tag{2.6}$$

If U is a quadratic form in ε_{ij}, then

$$B^{-1} = {}^1/_2 \min (B_2^{-1},\ 2B_3^{-1})$$

In the case of a homogeneous isotropic body

$$U = \frac{1}{2}\left[\frac{\lambda}{\mu}(\varepsilon + \varepsilon_\alpha{}^\alpha)^2 + 2\varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} + 2\varepsilon^2 + 4\varepsilon_\alpha\varepsilon^\alpha\right]$$

hence we can set

$$A_1 = 2\alpha, \quad A_2 = 2 + \frac{\lambda}{\mu}\frac{1-\alpha}{\lambda/\mu + 1 - \alpha} = 2\,\frac{1+\nu-(1-\nu)\alpha}{1-(1-2\nu)\alpha}, \quad A_3 = 2$$

$$B_1 = 2\max\left\{1, 1 + \beta\frac{\lambda}{\mu}\right\}, \quad B_2 = 2 + \frac{\lambda}{\mu}\frac{\beta}{\beta - 1} = 2\left[1 + \frac{\nu}{1-2\nu}\frac{\beta}{\beta-1}\right], \quad B_3 = 2$$

$$(0 < \alpha \leqslant 1, \quad 1 < \beta < +\infty)$$

Here λ, μ are Lamé coefficients, ν is the Poisson's ratio, α, β are arbitrary parameters varying within mentioned limits. In particular, for $\alpha = 1$, $\beta = {}^3/_2$, $\nu \geqslant 0$ we have

$$A_1 = A_2 = A_3 = 2, \qquad B_1 = B_2 = 2 + 3\frac{\lambda}{\mu}, \qquad B_3 = 2$$

$$B = 4\left(1 + \frac{3\lambda}{2\mu}\right)$$

3. The constants b and b_n. The inequality (1.6) yields a lower bound of the constant γ in terms of the natural vibrations frequency of an elastic cylinder. Further we consider the estimates of γ in terms of other body characteristics, including the constants b and b_n.

The constant b. Let us consider the vector field w^i in some bounded domain V For all the vector fields w^i satisfying the constraints

$$\int_V w^i d\tau = 0, \qquad \int_V (w_{i,j} - w_{j,i})\, d\tau = 0 \tag{3.1}$$

the inequality

$$b\int_\Omega w_i w^i d\sigma \leqslant \int_V U d\tau \tag{3.2}$$

holds, where Ω is part of the boundary ∂V of the domain V, and the constant b is the best constant in the inequality (3.2). Evidently b depends only on V, Ω and the elastic moduli.

The inequality (3.2) follows from (2.3) and the inequalities (3.3) of Poincaré [3, 8], (3.4) of Ehrlich [9], and (3.5) of Korn [10 – 13]:

$$\Lambda_P{}^2\int_V u^2 d\tau \leqslant \int_V u_{,i} u^{,i} d\tau, \qquad \int_V u\, d\tau = 0 \tag{3.3}$$

$$\int_{\partial V} u^2 d\sigma \leqslant A\left(\int_V u_{,i} u^{,i} d\tau + \int_V u^2 d\tau\right) \tag{3.4}$$

$$\int_V w_{i,j} w^{i,j} d\tau \leqslant K \int_V U d\tau \tag{3.5}$$

The Korn inequality is satisfied for functions satisfying the second constraint in (3.1).

Notes. 1. If V is a cylinder of height h and Ω is the base of the cylinder, then

$$b \to 0 \quad \text{for} \quad h \to 0 \quad \text{and} \quad h \to +\infty \tag{3.6}$$

In order to see this, let us consider the displacement field (the cylinder axis is parallel to the x-axis and the cylinder base Ω lies in the $x = 0$ plane)

$$w = u_0 \cos \frac{2\pi x}{h}, \quad w^\alpha = 0, \quad u_0 = \text{const}, \quad h \to +\infty \tag{3.7}$$

$$w = u(x^\alpha), \quad w_\alpha = -u_{,\alpha}\left(x - \frac{h}{2}\right), \quad \int_\Omega u(x^\alpha)\, d\sigma = 0 \tag{3.8}$$

$$\int_\Omega u_{,\alpha} d\sigma = 0, \quad h \to 0$$

The conditions (3.1) are evidently satisfied for the displacement fields (3.7) and (3.8). Substituting (3.7) and (3.8) into (3.2) and replacing U by a large quadratic form according to (2.3), we obtain the upper bound for b

$$b \leqslant B_2 \pi^2 / h, \quad h \to \infty$$

$$b \leqslant \frac{B_1 h^3}{24} \int_\Omega u_{,\alpha\beta} u^{,\alpha\beta} d\sigma \Big/ \int_\Omega \left(u^2 + \frac{h^2}{4} u_{,\alpha} u^{,\alpha}\right) d\sigma, \quad h \to 0$$

from which (3.6) follows.

2. The property (3.6) of b permits the expectation that a domain with maximum value of b can be selected from all domains V with fixed base Ω.

The constant b_n. Let b_n denote the best constant in the inequality

$$b_n \int_\Omega w_n^2 d\sigma \leqslant \int_V U d\tau, \qquad w_n = w^i n_i \tag{3.9}$$

Here n_i are components of the vector normal to the boundary of the domain V. The field of the displacement vectors must still be subject to conditions excluding solid motion. It is clear that these conditions should be less than for the inequality (3.2). For example, if Ω is a domain in the plane $x = \text{const}$, then the constraints

$$\int_V (w_{,\alpha} - \partial w_\alpha / \partial x)\, d\tau = 0, \qquad \int_V w d\tau = 0 \tag{3.10}$$

can be taken as such conditions.

Note. Analysis of the inequality (3.9) in the displacement fields (3.7) and (3.8) shows that for b_n the property (3.6) holds.

4. Estimate of the energy decay rate in terms of the constant b. Let external forces applied to Ω_0 cause surface loads p^i at a section of the body $\Omega(x)$. Let $V'(x)$ denote a subdomain of the domain $V(x)$ with the base $\Omega(x)$, for which the constant b is known and takes on the greatest possible value, and let us examine the strain of the body $V'(x)$ by the surface forces p^i. According to (2.2), (3.2) and the equilibrium equations $p_{,j}^{ij} = 0$, $p^{ij} n_j = p_i$, we have for the elastic energy E' of the body $V'(x)$

$$E' = \int_{V'} U d\tau \leqslant \int_{V'} \frac{\partial U}{\partial \varepsilon_{ij}} \varepsilon_{ij} d\tau = \int_{V'} p^{ij} w_{i,j} d\tau = \int_{\Omega(x)} p^i w_i dx^1 dx^2 \quad (4.1)$$

If a vector corresponding to the solid motion is added to the displacement vector in (4.1), then the left side of the inequality does not change since the elastic energy is invariant with respect to translations and rotations, and the right side does not change because the load is self-balanced in $\Omega(x)$. We use this arbitrariness to satisfy the constraint (3.1). By the Cauchy-Buniakowski inequality

$$\int_\Omega p^i w_i d\sigma \leqslant \left(\int_\Omega p_i p^i d\sigma \right)^{1/2} \left(\int_\Omega w_i w^i d\sigma \right)^{1/2} \leqslant \left(\int_\Omega p_i p^i d\sigma \right)^{1/2} (b^{-1} E')^{1/2}$$

Hence

$$bE' \leqslant \int_\Omega p_i p^i d\sigma \quad (4.2)$$

On the other hand (see [14], a more general assertion is proved in the supplement) $E \leqslant E'$. Using (2.6), we obtain from (4.2)

$$bE \leqslant \int_\Omega p_i p^i d\sigma \leqslant B \int_\Omega U d\sigma \quad (4.3)$$

The inequality (4.3) results in the following estimate for γ:

$$b / B \leqslant \gamma \quad (4.4)$$

Note. For an isotropic body with $\lambda \geqslant 0$ the constant B is given by the formula

$$B = 4\left(1 + \frac{\lambda}{2\mu} \frac{\beta}{\beta - 1}\right)$$

where β is an arbitrary number between 1 and $+\infty$. Since b and γ are independent of β, then $\inf B(\beta) = 4(1 + \lambda / 2\mu)$ can be taken as B. It follows from this formula that the estimate (4.4) is degraded for larger λ / μ, since $B \to \infty$ as $\lambda / \mu \to \infty$.

This is apparently not related to the crux of the matter. Let us note that the estimate (1.6) possesses the same property. In fact,

$$1 / (\rho \omega_0^2) \leqslant R / \mu_m$$

can be written for the fundamental natural vibrations frequency, where the constant R is independent of the elastic moduli and is determined only by the body geometry. Hence

$$s(l) \leqslant R^{1/2} \mu_M / \mu_m$$

and the estimate (1.6) corresponds to the inequality

$$R^{-1/2} \mu_m / \mu_M \leqslant \gamma$$

For an isotropic body $\mu_m = 2\mu$, $\mu_M = 3\lambda + 2\mu$, and the estimate for γ is degraded as λ / μ increases.

5. Estimation of the energy decay rate in terms of b_n. The problem of calculating b, exactly as the problem of calculating ω_0 in the estimate (1.6), are complex problems. In this connection, the estimation of γ in term of b_n is of interest since it is simpler to seek the constant b_n than b.

In order to obtain more accurate estimates, we assume that the body is physically linear (U is a quadratic form in the strain tensor components). We understand $V'(x)$ to be a subdomain of $V(x)$ for which the constant b_n is known and takes on a possibly larger value.

Just as in Sect. 4, we have

$$E' = \frac{1}{2}\int_{V'(x)} \frac{\partial U}{\partial \varepsilon_{ij}} \varepsilon_{ij}\, d\tau = \frac{1}{2}\int_{\Omega(x)} p^i w_i d\sigma = -\frac{1}{2}\int_{\Omega} (pw + p^\alpha w_\alpha)\, d\sigma \quad (5.1)$$

We use the arbitrariness in the selection of the displacement vector in order to satisfy the constraints (3.10) as well as the relationships

$$\int_\Omega w_\alpha d\sigma = 0, \qquad \int_\Omega (w_{\alpha,\beta} - w_{\beta,\alpha})\, d\sigma = 0 \quad (5.2)$$

The inequality

$$\Lambda^2_e \int_\Omega w_\alpha w^\alpha d\sigma \leqslant \int_\Omega \varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} d\sigma \quad (5.3)$$

holds for two-dimensional vector fields subjected to the conditions (5.2). It follows from the two-dimensional analogs of the inequalities (3.3) and (3.5).

By using (3.9), (5.3) and the Cauchy-Buniakowski inequalities, we find from (5.1)

$$2E' \leqslant \left(\int_\Omega p^2 d\sigma\right)^{1/2}\left(\int_\Omega w^2 d\sigma\right)^{1/2} + \left(\int_\Omega p_\alpha p^\alpha d\sigma\right)^{1/2}\left(\int_\Omega w_\alpha w^\alpha d\sigma\right)^{1/2} \leqslant \quad (5.4)$$

$$\left(2B_2\int_\Omega U d\sigma\right)^{1/2}(b_n^{-1}E')^{1/2} + \left(B_3\int_\Omega U d\sigma\right)^{1/2}\left(\Lambda_e^{-2}\int_\Omega 2A_1^{-1} U d\sigma\right)^{1/2} \leqslant$$

$$\alpha E' + \left(\frac{1}{2}\alpha^{-1}B_2 b_n^{-1} + (2B_3A_1^{-1})^{1/2}\Lambda_e^{-1}\right)\int_\Omega U d\sigma$$

where α is an arbitrary positive parameter. From (5.4)

$$E \leqslant E' \leqslant (2-\alpha)^{-1}({}^1/_2\alpha^{-1}B_2 b_n^{-1} + (2B_3A_1^{-1}\Lambda_e^{-2})^{1/2})\int_\Omega U d\sigma$$

Minimizing the factor in the right side with respect to α and substituting the result in (1.1), we arrive at the following lower bound for γ:

$$\frac{8b_n}{B_2}\left[1 + \left(1 + \frac{4b_n}{B_2\Lambda_e}\left(\frac{2B_3}{A_1}\right)\right)^{1/2}\right]^{-2} \leqslant \gamma \quad (5.5)$$

Both b_n and Λ_e in the left side of (5.5) depend on x.

Notes. 1. For an isotropic body, we should put $B_3 = 2$ in (5.5), should take the maximum value $A_1 = 2$ and the minimum value, $B_2 = 2 + \lambda/\mu$ since the right side of (5.5) is a decreasing function in B_2 and a growing function in A_1

$$\frac{4b_n(1-2\nu)}{1-\nu}\left[1 + \left(1 + \frac{2\sqrt{2}\, b_n(1-2\nu)}{(1-\nu)\Lambda_e}\right)^{1/2}\right]^{-2} \leqslant \gamma \quad (5.6)$$

2. The constant Λ_e can be estimated in terms of the Poincaré constant (3.3) and the Korn constant K of the domain Ω

$$\Lambda_e^{-2} \leqslant \Lambda_P^{-2} K \quad (5.7)$$

The best constant in the inequality

$$\int_\Omega w_{\alpha,\beta} w^{\alpha,\beta} d\sigma \leqslant K \int_\Omega \varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} d\sigma \quad (5.8)$$

is here understood to be the Korn constant. The functions w_α in (5.8) satisfy the second constraint in (5.2). Using (5.7), it is possible to rewrite (5.6) as

$$\frac{4(1-2\nu)\,b_n}{1-\nu}\left[1+\left(1+\frac{2\sqrt{2K}\,b_n(1-2\nu)}{(1-\nu)\Lambda_P}\right)^{1/2}\right]^{-2}\leqslant\Upsilon \tag{5.9}$$

6. The estimate b_n for a rod. Let us consider a cylindrical rod of height h $(0\leqslant x\leqslant h)$ with cross section Ω. We take the origin of the coordinate system x^α at the center of gravity of the cross section. We introduce the notation

$$u(x^\alpha)=\int_0^h wx(h-x)\,dx\Big/\int_0^h x(h-x)\,dx \tag{6.1}$$

$$\psi^\alpha(x^\beta)=\int_0^h w^\alpha\left(x-\frac{h}{2}\right)dx\Big/\int_0^h\left(x-\frac{h}{2}\right)^2dx$$

In constructing the estimate b_n it is convenient to impose the constraints

$$\int_\Omega u\,d\sigma=0,\qquad\int_\Omega(u_{,\alpha}-\psi_\alpha)\,d\sigma=0 \tag{6.2}$$

instead of the constraints (3.10) on the displacement vector components.

The preceding reasoning is evidently independent of the kind of constraints excluding the solid motion.

6.1. We prove the inequality

$$\int_\Omega w_0^2\,d\sigma\leqslant(1+\alpha)\int_\Omega u^2\,d\sigma+\frac{13}{35}(1+\alpha^{-1})\,h\int_\Omega\int_0^h\varepsilon^2\,d\sigma dx \tag{6.3}$$

where $w_0=w(0,\ x^\alpha)$ and α is an arbitrary positive number.

In fact

$$w(x,x^\alpha)-w_0=\int_0^x\varepsilon\,dx,\qquad\varepsilon\equiv\frac{\partial w}{\partial x}$$

Multiplying by $6(h-x)x/h^3$ and integrating with respect to x, we obtain

$$w_0=u-6h^{-3}\int_0^h(h-x)\,x\left(\int_0^x\varepsilon\,d\xi\right)dx=u-h\int_0^1(1-3\xi^2+2\xi^3)\,\varepsilon(h\xi,x^\alpha)\,d\xi$$

Squaring both sides and using the Cauchy-Buniakowski inequalities, we arrive at the relationship

$$w_0^2\leqslant(1+\alpha)\,u^2(x^\alpha)+(1+\alpha^{-1})\,\frac{13h}{35}\int_0^h\varepsilon^2\,dx$$

which yields (6.3) when integrated over the domain Ω.

6.2. It has been shown in [15] that the elastic energy of the Reissner model yields an accurate lower bound for the elastic energy of a cylindrical body (plate or rod). Let us write the appropriate inequality in application to the case under consideration as

$$I\leqslant\frac{1}{2}\int_\Omega\int_0^h[A_1\varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta}+2A_3\varepsilon_\alpha\varepsilon^\alpha]\,d\sigma dx \tag{6.4}$$

$$I = \frac{1}{2}\int_{\Omega}\left[A_1\frac{h^3}{12}\psi_{(\alpha,\beta)}\psi^{(\alpha,\beta)} + A_3\frac{5h}{12}(u_{,\alpha} + \psi_{\alpha})(u^{,\alpha} + \psi^{\alpha})\right]d\sigma$$

6.3. Let m denote the best constant in the inequality

$$\int_{\Omega} u^2 d\sigma \leqslant mI \tag{6.5}$$

Here u and ψ_{α} satisfy the constraints (6.2).

6.4. From (6.3) – (6.5) we obtain

$$\int_{\Omega} w_0{}^2 d\sigma \leqslant (1+\alpha)\, m\cdot\frac{1}{2}\int_{\Omega}\int_0^h [A_1\varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} + 2A_3\varepsilon_{\alpha}\varepsilon^{\alpha}]\, d\sigma dx +$$

$$\frac{13}{35}(1+\alpha^{-1})\, h\cdot 2A_2^{-1}\int_{\Omega}\int_0^h \frac{1}{2}A_2\varepsilon^2 d\sigma dx \leqslant$$

$$\max\left\{(1+\alpha)\, m,\ \frac{26h}{35A_2}(1+\alpha^{-1})\right\}\int_{\Omega}\int_0^h U d\sigma dx$$

Minimizing the coefficient in the right side with respect to α, we arrive at the following bound:

$$\left(m + \frac{26h}{35A_2}\right)^{-1} \leqslant b_n \tag{6.6}$$

There remains to estimate the constant m in terms of more accessible characteristics of the cross section.

6.5. Let $\varkappa$ denote the best constant in the inequality

$$\left(\int_{\Omega} u_{,\alpha}x_{\beta}e^{\alpha\beta}d\sigma\right)^2 \leqslant \varkappa\int_{\Omega} u_{,\alpha}u^{,\alpha}d\sigma\cdot\int_{\Omega} x_{\alpha}x^{\alpha}d\sigma \tag{6.7}$$

Here $e^{\alpha\beta}$ is the two-dimensional Levi-Civita symbol. Evidently $\varkappa = 0$ when Ω is a circle.

6.6. Let Λ denote the best constant in the inequality

$$\Lambda^2\int_{\Omega} u^2 d\sigma \leqslant \int_{\Omega}\left(u_{,\alpha}u^{,\alpha} + \frac{3}{1-\varkappa}c_{\alpha}c^{\alpha}\right)d\sigma \tag{6.8}$$

where

$$\int_{\Omega} u d\sigma = 0, \qquad \Omega c_{\alpha} = \int_{\Omega} u_{,\alpha}d\sigma \tag{6.9}$$

The letter Ω also denotes the area of the domain Ω.. Evidently

$$\Lambda^{-2} \leqslant \Lambda_P{}^{-2} \tag{6.10}$$

6.7. Let us make the substitution $\psi^{\alpha} \to \psi'^{\alpha}$ in the inequality (6.5)

$$\psi^{\alpha} = c^{\alpha} + \omega e^{\alpha\beta}\, x_{\beta} + \psi'^{\alpha} \tag{6.11}$$

$$\omega = \frac{1}{2\Omega}\int_{\Omega} e^{\alpha\beta}\psi_{\alpha,\beta}d\sigma \tag{6.12}$$

where c^{α} is defined by the formula (6.9).

According to the second constraint in (6.2) and (6.11), (6.12), the functions ψ'^{α} satisfy the condition

$$\int_{\Omega} \psi'^{\alpha} d\sigma = 0, \quad \int_{\Omega} (\psi'_{\alpha,\beta} - \psi'_{\beta,\alpha})\, d\sigma = 0 \tag{6.13}$$

Hence, by virtue of (5.3)

$$\int_{\Omega} \psi_{(\alpha,\beta)}\psi^{(\alpha,\beta)} d\sigma = \int_{\Omega} \psi'_{(\alpha,\beta)}\psi'^{(\alpha,\beta)} d\sigma \geqslant \Lambda_e^2 \int_{\Omega} \psi_{\alpha}'\psi'^{\alpha} d\sigma \tag{6.14}$$

Therefore

$$I \geqslant \frac{5hA_3}{24}\int_{\Omega}\left[\frac{B_1h^2\Lambda_e}{5A_3}\psi_{\alpha}'\psi'^{\alpha} + (u_{,\alpha} + c_{\alpha} + \omega e_{\alpha\beta}x^{\beta} + \psi_{\alpha}')^2\right] d\sigma \geqslant \tag{6.15}$$

$$A\int_{\Omega}(u_{,\alpha} + c_{\alpha} + \omega e_{\alpha\beta}x^{\beta})^2\, d\sigma$$

$$A = \frac{5A_1A_3h^3\Lambda_e}{24\,(5A_3 + A_1h^2\Lambda_e^2)}$$

Minimizing the right side of (6.15) with respect to ω and using (6.7) and (6.8), we obtain

$$I \geqslant A\left[\int_{\Omega}(u_{,\alpha}u^{,\alpha} + 3c_{\alpha}c^{\alpha})\, d\sigma - \left(\int_{\Omega} u_{,\alpha}x_{\beta}e^{\alpha\beta}d\sigma\right)^2 \Big/ \int_{\Omega} x_{\alpha}x^{\alpha}d\sigma\right] \geqslant$$

$$A\,(1-\varkappa)\int_{\Omega}\left(u_{,\alpha}u^{,\alpha} + \frac{3}{1-\varkappa}c_{\alpha}c^{\alpha}\right)d\sigma \geqslant A\,(1-\varkappa)\,\Lambda^2\int_{\Omega} u^2 d\sigma$$

We hence find

$$m \leqslant A^{-1}(1-\varkappa)^{-1}\ \Lambda^{-2} \tag{6.16}$$

$$\frac{1}{2}\left(\frac{12\,(5A_3 + B_1h^2\Lambda_e^2)}{5A_1A_3h^3\Lambda_e^2\Lambda^2\,(1-\varkappa)} + \frac{13h}{35A_2}\right)^{-1} \leqslant b_n \tag{6.17}$$

Maximizing the left side of (6.17) with respect to h results in a relationship of the form

$$\Lambda_P\varphi\left(\frac{\Lambda}{\Lambda_P}, \frac{\Lambda_e}{\Lambda_P}, \varkappa\right) \leqslant b_n \tag{6.18}$$

Note. An essential property of the constant b_n results from (6.18): b_n can be bounded from below by a quantity which diminishes λ-fold as the cylinder cross section is stretched λ-fold. This follows from the fact that the constants Λ, Λ_e, Λ_P acquire a factor $1/\lambda$, while the constant $\varkappa$ does not vary as the cross section is stretched. According to (5.5), γ possesses an analogous property.

7. Estimation of the rate of energy decrease in a semi-infinite circular isotropic homogeneous rod. For a circular rod $\varkappa = 0$ and the Poincaré and Korn constants are known

$$\Lambda_P = jr^{-1}, \qquad K = 4$$

Here $j = 1.845$ is the first zero of the derivative of the Bessel function $J_1(x)$ and r is the cross-sectional radius. The Korn constant has been calculated in [16].

In the case of a homogeneous isotropic rod $A_3 = 2$ and the constants A_1 and A_2 can vary as functions of the parameter α (see Sect. 2), maximizing the left side of (6.17) with respect to α. For simplicity we set $\alpha = 1$, $A_1 = A_2 = 2$. Moreover, in conformity with (6.10), let us replace the constant Λ^{-2} by Λ_P^{-2}. We finally obtain the following estimate of b_n in terms of Λ_P, Λ_e, h:

$$\frac{1}{2}\left[\frac{6(5+h^2\Lambda_e^2)}{5h^3\Lambda_e^2\Lambda_P^2}+\frac{13h}{70}\right]^{-1}\leqslant b_n$$

or in terms of Λ_P, K and h

$$\frac{1}{2}\left[\frac{6(5K+h^2\Lambda_P^2)}{5h^3\Lambda_P^4}+\frac{13h}{70}\right]^{-1}\leqslant b_n \qquad (7.1)$$

Maximizing the left side with respect to h, we obtain

$${}^1/_7\Lambda_P\leqslant b_n \qquad (7.2)$$

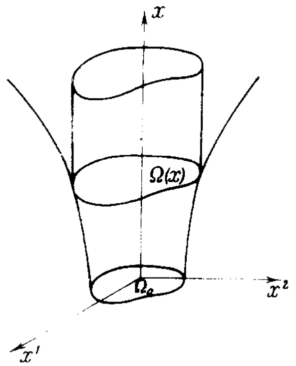

Fig. 2

The inequalities (5.9) and (7.1) yield the lower bound of the decay rate for an arbitrary load self-balanced at the endface, in particular

$$0.35r^{-1}\leqslant\gamma,\quad \nu=0;\qquad 0.27r^{-1}\leqslant\gamma,\qquad \nu={}^1/_4$$
$$0.22r^{-1}\leqslant\gamma,\quad \nu={}^1/_3$$

In conformity with the Note in Sect. 4, the estimate is degraded as $\nu={}^1/_2\,(1+\mu/\lambda)^{-1}$ increases. The energy decay rate for an arbitrary load is unknown. The decay rate for an axisymmetric load applied to the endface is calculated in a number of papers (see [17, 18]). The corresponding results can be considered an upper bound for γ

$$\gamma\leqslant 5r^{-1},\quad \nu=0;\qquad \gamma\leqslant 5.4r^{-1},\quad \nu={}^1/_4$$

8. Energy decay rate in cone-type bodies. Let us consider bodies whose sections by the planes $x=\text{const}$ are similar (see Fig. 2), i.e.

$$\Omega(x)=\{x,\ x^\alpha : x^\alpha=\lambda(x)x_0^\alpha,\ x_0^\alpha\in\Omega_0\},\ \lambda(0)=1$$

Let us also assume that $\lambda(x)$ is a nondecreasing function of x. Then a cylinder with base $\Omega(x)$ can be taken as the subdomain $V'(x)$ for the domain $V(x)$, and the constant b_n of the appropriate cylinder as the constant b_n in (5.9). According to (5.9) and the Note in Sect. 6

$$b_n(0)/\lambda(x)\leqslant b_n(x),\qquad \gamma(0)/\lambda(x)\leqslant\gamma(x)$$

Therefore, for such bodies

$$E(x)\leqslant E(0)\exp\left[-\gamma(0)\int_0^x\frac{dx}{\lambda(x)}\right]$$

In the case of a conical body whose apex is at a point with the coordinates $(-x_0,\ 0,\ 0)$, $\lambda(x)=1+x/x_0$, hence

$$E(x)\leqslant E(0)(1+x/x_0)^{-x_0\gamma(0)} \qquad (8.1)$$

As $x_0\to\infty$ the cone is converted into a cylinder and the estimate (8.1) goes over into the exponential decay law (1.5).

9. Supplement. About the elastic energy of bonded bodies. The total elastic energy diminishes when a body, deformed by certain surface forces, is bonded to an undeformed body along a load-free surface. Some extensions are formulated below.

Let us examine the problem about the minimum of the functional

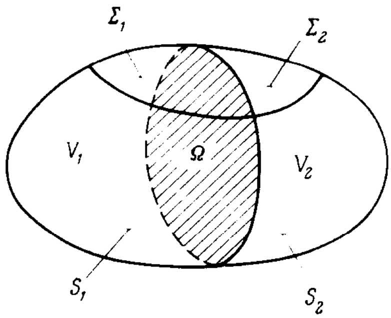

Fig. 3

$$I = E - \int_V F_\alpha u^\alpha dx - \int_{\partial V} f_\alpha u^\alpha d\sigma$$

$$dx = dx^1 \ldots dx^n$$

$$E = \int_V U\left(x^i, u^\alpha, \frac{\partial u^\alpha}{\partial x^i}\right) dx$$

The Greek superscripts here take on the values $1, \ldots, m$, and the Latin superscripts the values $1, \ldots, n$, V is the domain of n-space, F_α, f_α are given functions of x^i. The minimum is sought over all functions u^α taking the given values

$$u^\alpha = \varphi^\alpha \quad \text{on } \Sigma$$

on the part Σ of the boundary ∂V of the domain V

Let the domain V be divided into the subdomains V_1 and V_2 with the boundaries ∂V_1 and ∂V_2. By Σ_1 and Σ_2, S_1 and S_2 we denote the parts of the surfaces Σ and ∂V belonging to ∂V_1 and ∂V_2, respectively, by Ω we denote the hypersurface separating V_1 and V_2 (see Fig. 3), and we form the functionals (p_α are arbitrarily assigned functions)

$$I_1 = E_1 - \int_{V_1} F_\alpha u^\alpha dx - \int_{S_1} f^\alpha u^\alpha d\sigma - \int_\Omega p_\alpha u^\alpha d\sigma$$

$$I_2 = E_2 - \int_{V_2} F_\alpha u^\alpha dx - \int_{S_2} f_\alpha u^\alpha d\sigma + \int_\Omega p_\alpha u^\alpha d\sigma$$

$$E_1 = \int_{V_1} U dx, \qquad E_2 = \int_{V_2} U dx, \qquad I = I_1 + I_2$$

Since the sum of the minima does not exceed the minimum of the sum

$$\inf_{u^\alpha} I = \inf_{u^\alpha} (I_1 + I_2) \geqslant \inf_{u^\alpha} I_1 + \inf_{u^\alpha} I_2$$

we obtain (the minimal values of the functionals are marked with the zero superscript)

$$I_1^\circ + I_2^\circ \leqslant I^\circ \tag{9.1}$$

9.1. We consider two linearly elastic bodies V_1 and V_2 deformed by surface and volume forces. Some sections Σ_1 and Σ_2 of the surfaces of the bodies V_1 and V_2 can be rigidly clamped ($u^\alpha|_{\Sigma_1, \Sigma_2} = 0$). Let the boundaries ∂V_1 and ∂V_2 of the bodies V_1 and V_2 contain the surfaces Ω_1 and Ω_2 representing two sides of some surface Ω upon superposition, and let forces acting at points corresponding to one another during superposition be equal and opposite in direction. Then the sum of the elastic energies of the bodies V_1 and V_2 is less than the elastic energy of a body obtained as a result of bonding V_1 and V_2 along Ω. (The bodies are said to be bonded along Ω if the u^α are continuous upon going across Ω.).

In fact, the functionals E, E_1 and E_2 are quadratic and have the meaning of an energy in linear elasticity theory. Under the rigid clamping condition

$$I^\circ = -E^\circ, \qquad I_1^\circ = -E_1^\circ, \qquad I_2^\circ = -E_2^\circ.$$

The assertion formulated results from the inequality (9.1).

9.2. Let us consider two elastic bodies V_1 and V_2 deformed by displacements given on the surfaces Σ_1 and Σ_2. The boundaries ∂V_1 and ∂V_2 of the bodies V_1 and V_2 contain the surfaces Ω_1 and Ω_2, which represent the superposition of two sides of some surface Ω. Then the sum of the elastic energies of the bodies V_1 and V_2 is greater than the elastic energy of the body obtained as a result of bonding the bodies V_1 and V_2 along Ω.

This assertion follows directly from (9.1) since $I^\circ = E^\circ$, $I = E_1^\circ$ and $I_2^\circ = E_2^\circ$ in the case under consideration.

Notes. 1. If the bodies V_1 and V_2 are deformed by surface and volume forces and by nonzero displacements given on the boundary, then the energy of the bonded bodies can be both greater as well as less than the sum of the energies of the bodies prior to bonding.

2. The property of convexity of U was not used in deriving (9.1). The proof of the theorem about the bonding of elastic bodies is based on the Castigliano principle [14] for whose construction the convexity of U is essential [19]. Let us note that in the geometrically nonlinear theory of elasticity U is not a convex function of the displacement gradient.

3. With some terminology changes (see footnote, p. 802) the assertions (9.1) and (9.2) go over into a number of other mechanics problems including the problem of the steady potential flow of an ideal incompressible fluid and the problem of steady heat conduction. These assertions permit reduction of the elastic energy estimation (or of the kinetic energy and dissipation, respectively) for a body with complex geometric outlines to the estimation for a body with a simple geometric shape.

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INHOMOGENEOUS LAYER BONDED TO A HALF-SPACE UNDER THE ACTION OF INTERNAL AND EXTERNAL FORCES

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We obtain a solution of the problem on the stress-strain state of an inhomogeneous isotropic layer, the elastic characteristics of which are bounded and integrable functions of a single Cartesian coordinate. The layer is bonded continuously to a homogeneous half-space, and is acted upon by the mass forces.

The problem arises in the analysis of coverings. The earlier papers dealt with particular cases in which an open surface was acted upon by a normal load. In [1, 2] such a problem was studied for an exponential law of variation of the modulus of elasticity with depth, with the Poisson's ratio remaining constant, while in [3] the same problem was studied for a hyperbolic law. This was done by considering an axisymmetric deformation of an inhomogeneous layer resting on a perfectly rigid support. In [4] a solution was obtained for an incompressible material in which the shear modulus changed linearly with depth, while in [5, 6] a solution was obtained for an arbitrary law of change and a constant Poisson's ratio. The method used in the latter case deserves attention. In the course of solution the layer was replaced by a system of n interconnected homogeneous isotropic plates of equal thickness, the elasticity moduli of which were defined by a given function of the inhomogeneity. Passage to the limit $n \to \infty$ gave a formally exact solution of the initial problem. It appears, that the action of shearing loads and forces applied within the layer has, so far, not been investigated.

1. Using the method developed in [7], we split the system of equilibrium equations